

Prvi kolokvijum iz Matematike 1, Hemija, 18.12.2011. GRUPA I

- 1.(4 poena) Ispitati konvergenciju reda $\sum \frac{2}{3^n} \left(\frac{2n-1}{n+2}\right)^n$
- 2.(4 poena) Odrediti sve lokalne ekstremume funkcije $f(x) = e^{-x}\sqrt{x^2-2}$
- 3.(4 poena) Odrediti kosu asimptotu funkcije $f(x) = \sqrt{x^2+5x-6}$ kad $x \rightarrow \infty$
- 4.(4 poena) Izračunati $\lim_{x \rightarrow 0} \frac{\sin^3(2x)}{x^3}$
- 5.(4 poena) Izračunati $\int \frac{x^2+4x-4}{(x-1)(x^2-4)} dx$
- 6.(5 poena) Izračunati $\int \frac{x}{\cos^2 x} dx$

Prvi kolokvijum iz Matematike 1, Hemija, 18.12.2011. GRUPA II

- 1.(4 poena) Ispitati konvergenciju reda $\sum \frac{5}{2^n} \left(\frac{3n+2}{n-2}\right)^n$
- 2.(4 poena) Odrediti sve lokalne ekstremume funkcije $f(x) = x^4 \ln x^2$
- 3.(4 poena) Odrediti kosu asimptotu funkcije $f(x) = 3x + \sqrt{x^2+4x}$ kad $x \rightarrow \infty$
- 4.(4 poena) Izračunati $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(4x) \cos(2x)}{(x-\frac{\pi}{4})^2}$
- 5.(4 poena) Izračunati $\int \frac{x^2+2}{x^3-x} dx$
- 6.(5 poena) Izračunati $\int x^2 \sin 2x dx$

Rešenje:

1. grupa I

Primenom Košijevog kriterijuma:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{2}{3^n} \left(\frac{2n-1}{n+2} \right)^n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{2}}{3} \frac{2n-1}{n+2} = (*)$$

Podelimo brojilac i imenilac sa n i iskoristimo $\lim_{n \rightarrow \infty} \sqrt[n]{2} = 1$

$$(*) = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3 + \frac{6}{n}} = \frac{2}{3}$$

Kako je $\frac{2}{3} < 1$, po Košijevom kriterijumu red konvergira.

1. grupa II

Primenom Košijevog kriterijuma:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{5}{2^n} \left(\frac{3n+2}{n-2} \right)^n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{5}}{2} \frac{3n+2}{n-2} = (*)$$

Podelimo brojilac i imenilac sa n i iskoristimo $\lim_{n \rightarrow \infty} \sqrt[n]{5} = 1$

$$(*) = \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{2 - \frac{4}{n}} = \frac{3}{2}$$

Kako je $\frac{3}{2} > 1$, po Košijevom kriterijumu red divergira.

2. grupa I

$$f(x) = e^{-x} \sqrt{x^2 - 2}$$

Domen funkcije: $x^2 - 2 \geq 0 \Rightarrow x^2 \geq 2 \Rightarrow x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

$$f'(x) = -e^{-x} \sqrt{x^2 - 2} + e^{-x} \frac{1}{2\sqrt{x^2 - 2}} 2x = e^{-x} \left(-\sqrt{x^2 - 2} + \frac{x}{\sqrt{x^2 - 2}} \right) = e^{-x} \left(\frac{-x^2 + x + 2}{\sqrt{x^2 - 2}} \right)$$

Tražimo nule prvog izvoda: $f'(x) = 0 \Leftrightarrow -x^2 + x + 2 = 0 \Rightarrow x_1 = 2, x_2 = -1$

$x = -1$ ne pripada domenu funkcije, tako da je jedini kandidat za lok.extr. $x = 2$.

Funkcija na $(-\infty, -\sqrt{2}]$ opada, na $[\sqrt{2}, 2]$ raste i na $[2, \infty)$ opada $\Rightarrow$ Funkcija dostiže maksimum za $x = 2$.

$$f(2) = \frac{\sqrt{2}}{e^2} \Rightarrow \text{MAX}(2, \frac{\sqrt{2}}{e^2})$$

2. grupa II

$$f(x) = x^4 \ln x^2$$

Domen funkcije: $x^2 > 0 \Rightarrow x \neq 0 \Rightarrow x \in (-\infty, 0) \cup (0, \infty)$

$$f'(x) = 4x^3 \ln x^2 + x^4 \frac{1}{x^2} 2x = 4x^3 \ln x^2 + 2x^3 = x^3 (4 \ln x^2 + 2)$$

Tražimo nule prvog izvoda: $f'(x) = 0 \Leftrightarrow x_1 = 0$ ili $4 \ln x^2 + 2 = 0 \Rightarrow x_1 = 0, x_2 = -e^{-\frac{1}{4}}, x_3 = e^{-\frac{1}{4}}$
 $x = 0$ ne pripada domenu funkcije, tako da su jedini kandidati za lok.extr. $x_2 = -e^{-\frac{1}{4}}, x_3 = e^{-\frac{1}{4}}$.

Funkcija na $(-\infty, -e^{-\frac{1}{4}}]$ opada, na $[-e^{-\frac{1}{4}}, 0)$ raste, na $(0, e^{-\frac{1}{4}})$ opada i na $(e^{-\frac{1}{4}}, \infty)$ raste $\Rightarrow$ Funkcija dostiže minimume za $x_2 = -e^{-\frac{1}{4}}$ i $x_3 = e^{-\frac{1}{4}}$.

$$f(-e^{-\frac{1}{4}}) = f(e^{-\frac{1}{4}}) = -\frac{1}{2e} \Rightarrow \text{MIN1}(-e^{-\frac{1}{4}}, -\frac{1}{2e}), \text{MIN2}(e^{-\frac{1}{4}}, -\frac{1}{2e})$$

3. grupa I

$$f(x) = \sqrt{x^2 + 5x - 6}$$

Jednačina kose asimptote $y=kx+n$.

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+5x-6}}{x} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{5}{x} - \frac{6}{x^2}} = 1$$

$$n = \lim_{x \rightarrow \infty} [f(x) - kx] = \lim_{x \rightarrow \infty} [\sqrt{x^2+5x-6} - x] \cdot \frac{\sqrt{x^2+5x-6}+x}{\sqrt{x^2+5x-6}+x} = \lim_{x \rightarrow \infty} \frac{x^2+5x-6-x^2}{\sqrt{x^2+5x-6}+x} = (\text{nakon deljenja brojioca i imenioca sa } x) = \lim_{x \rightarrow \infty} \frac{5-\frac{6}{x}}{\sqrt{1+\frac{5}{x}-\frac{6}{x^2}}+1} = \frac{5}{2}$$

Kosa asimptota: $y = x + \frac{5}{2}$.

3. grupa II

$$f(x) = 3x + \sqrt{x^2 + 4x}$$

Jednačina kose asimptote $y=kx+n$.

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{3x + \sqrt{x^2+4x}}{x} = \lim_{x \rightarrow \infty} 3 + \sqrt{1 + \frac{4}{x}} = 4$$

$$n = \lim_{x \rightarrow \infty} [f(x) - kx] = \lim_{x \rightarrow \infty} [3x + \sqrt{x^2+4x} - 4x] = \lim_{x \rightarrow \infty} [\sqrt{x^2+4x} - x] \cdot \frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} = \lim_{x \rightarrow \infty} \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} = (\text{nakon deljenja brojioca i imenioca sa } x) = \lim_{x \rightarrow \infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} = \frac{4}{2} = 2$$

Kosa asimptota: $y = 4x + 2$.

4. grupa I

$$\lim_{x \rightarrow 0} \frac{\sin^3(2x)}{x^3} = \frac{0}{0}$$

$$\stackrel{\text{LP}}{=} \lim_{x \rightarrow 0} \frac{3 \sin^2(2x) \cos(2x) \cdot 2}{3x^2} = \frac{0}{0}$$

$$\stackrel{\text{LP}}{=} \lim_{x \rightarrow 0} \frac{2 \cdot 2 \sin(2x) \cos(2x) \cdot 2 \cos(2x) + 2 \sin^2(2x) (-\sin(2x)) \cdot 2}{2x} = \lim_{x \rightarrow 0} \frac{4 \sin(2x) \cos(2x) - 2 \sin^3(2x)}{x} = \frac{0}{0}$$

$$\stackrel{\text{LP}}{=} \lim_{x \rightarrow 0} \frac{4 \cos(2x) \cdot 2 \cos(2x) + 4 \sin(2x) (-\sin(2x)) \cdot 2 - 6 \sin^2(2x) \cos(2x) \cdot 2}{1}$$

$$= \lim_{x \rightarrow 0} (8 \cos^2(2x) - 8 \sin^2(2x) - 12 \sin^2(2x) \cos(2x)) = 8$$

4. grupa II

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(4x) \cos(2x)}{(x - \frac{\pi}{4})^2} = \frac{0}{0}$$

$$\stackrel{\text{LP}}{=} \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos(4x) \cdot 4 \cos(2x) + \sin(4x) (-\sin(2x)) \cdot 2}{2(x - \frac{\pi}{4})} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \cos(4x) \cos(2x) - \sin(4x) \sin(2x)}{x - \frac{\pi}{4}} = \frac{0}{0}$$

$$\stackrel{\text{LP}}{=} \lim_{x \rightarrow \frac{\pi}{4}} \frac{2(-\sin(4x)) \cdot 4 \cos(2x) + 2 \cos(4x) (-\sin(2x)) \cdot 2 - \cos(4x) \cdot 4 \sin(2x) - \sin(4x) \cos(2x) \cdot 2}{1} = \lim_{x \rightarrow 0} \frac{6 \sin x \cos^2 x - 3 \sin^3 x}{1} =$$

$$\lim_{x \rightarrow \frac{\pi}{4}} (-10 \sin(4x) \cos(2x) - 8 \cos(4x) \sin(2x)) = -10 \cdot 0 \cdot 0 - 8 \cdot (-1) \cdot 1 = 8$$

5. grupa I

$$I = \int \frac{x^2+4x-4}{(x-1)(x^2-4)} dx = \int \frac{x^2+4x-4}{(x-1)(x-2)(x+2)} dx$$

$$\frac{x^2+4x-4}{(x-1)(x-2)(x+2)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2} = \frac{A(x^2-4) + B(x^2+x-2) + C(x^2-3x+2)}{(x-1)(x-2)(x+2)}$$

Formiramo sistem linearnih jednačina:

$$A + B + C = 1$$

$$B - 3C = 4$$

$$-4A - 2B + 2C = -4$$

čijim rešavanjem dobijamo: $A = -\frac{1}{3}, B = 2, C = -\frac{2}{3}$

$$I = -\frac{1}{3} \int \frac{dx}{x-1} + 2 \int \frac{dx}{x-2} - \frac{2}{3} \int \frac{dx}{x+2} = -\frac{1}{3} \ln|x-1| + 2 \ln|x-2| - \frac{2}{3} \ln|x+2| + c$$

5. grupa II

$$I = \int \frac{x^2+2}{x^3-x} dx = \int \frac{x^2+2}{x(x-1)(x+1)} dx$$

$$\frac{x^2+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

Formiramo sistem linearnih jednačina:

$$A + B + C = 1$$

$$B - C = 0$$

$$-A = 2$$

čijim rešavanjem dobijamo: $A = -2, B = \frac{3}{2}, C = \frac{3}{2}$
 $I = -2 \int \frac{dx}{x} + \frac{3}{2} \int \frac{dx}{x-1} + \frac{3}{2} \int \frac{dx}{x+1} = 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{3}{2}\ln|x+1| + c$

6. grupa I

$$\begin{aligned} \int \frac{x}{\cos^2 x} dx &= \left\{ \begin{array}{l} u = x, \quad du = dx \\ dv = \frac{dx}{\cos^2 x}, \quad v = \int \frac{dx}{\cos^2 x} = \operatorname{tg} x \end{array} \right\} \\ &= x \cdot \operatorname{tg} x - \int \operatorname{tg} x dx = x \cdot \operatorname{tg} x - \int \frac{\sin x}{\cos x} dx = \left\{ \begin{array}{l} t = \cos x \\ dt = -\sin x \end{array} \right\} \\ &= x \cdot \operatorname{tg} x - \int \frac{-dt}{t} = x \cdot \operatorname{tg} x + \ln|t| + C = x \cdot \operatorname{tg} x + \ln|\cos x| + C \end{aligned}$$

6. grupa II

$$\begin{aligned} \int x^2 \sin(2x) dx &= \left\{ \begin{array}{l} u = x^2, \quad du = 2x dx \\ dv = \sin(2x) dx, \quad v = \int \sin(2x) dx = \frac{1}{2} \cos(2x) \end{array} \right\} \\ &= x^2 \cdot \frac{-1}{2} \cos(2x) - \int 2x \cdot \frac{-1}{2} \cos(2x) dx = \\ &= -\frac{x^2}{2} \cos(2x) + \int x \cos(2x) dx = \left\{ \begin{array}{l} u = x, \quad du = dx \\ dv = \cos(2x) dx, \quad v = \int \cos(2x) dx = \frac{1}{2} \sin(2x) \end{array} \right\} \\ &= -\frac{x^2}{2} \cos(2x) + x \cdot \frac{1}{2} \sin(2x) - \int \frac{1}{2} \sin(2x) dx \\ &= -\frac{x^2}{2} \cos(2x) + \frac{x}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C = \frac{1-2x^2}{4} \cos(2x) + \frac{x}{2} \sin(2x) + C \end{aligned}$$